THE TEMPERED FRACTIONAL LAPLACIAN AS A SPECIAL CASE OF THE NONLOCAL LAPLACE OPERATOR

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Abstract. Tempered fractional operators provide an improved predictive capability for modeling anomalous effects that cannot be captured by standard partial differential equations. These effects include subdiffusion and superdiffusion, which often occur in, e.g., geoscience and hydrology. Tempered operators can be used in such models of heavy-tailed behavior while circumventing consequences of standard fractional models, such as divergent moments. In the first part of this work, we investigate the relationship between tempered and truncated fractional operators and the unified nonlocal diffusion operator, building upon the recently developed unified nonlocal calculus. In the second part of this work, with the purpose of finding a computationally cheap alternative to tempered fractional operators, we investigate the relationship between the (computationally expensive) tempered fractional Laplacian and the (computationally cheaper) truncated fractional Laplacian. Our main result shows the equivalence between truncated and tempered fractional energies and represents the first step towards the approximation of expensive fractional models with cheaper, but equivalent, alternatives.

1. Introduction. Fractional models can capture anomalous effects that standard partial differential equations (PDEs) fail to describe. In particular, they can model superdiffusion and subdiffusion processes; i.e. processes for which the mean square displacement is proportional to time to a fractional power, instead of being linear with respect to time, as is the case for PDEs. These operators have been used for decades in subsurface diffusion and transport, where the anomalous behavior is caused by heterogeneities in materials or media [1, 5, 6, 12, 11], and have also found application in turbulence [7, 8, 10] and, more recently, in machine-learning algorithms [13].

Fractional operators, such as the fractional Laplacian, are integral operators acting on the whole space and, as such, feature infinite interactions between points or domains. This fact allows one to model long range forces and reduces the regularity requirements on the solution. However, despite their improved predictive capabilities, fractional models come with a high computational cost due to the infinite range of interactions and the singularities in their kernels. In this work we focus on the former matter and investigate an equivalent alternative to tempered fractional operators that is computationally cheaper. The alternative of choice is *truncated fractional operators*; i.e. fractional operators whose range of interaction is limited to a ball of finite radius.

In the first part of this work we investigate the relationship between tempered and truncated fractional operators and the unified nonlocal Laplacian operator, introduced in [3]. Specifically, we investigate the composition of tempered and truncated fractional divergence and gradient and compare it with the tempered and truncated fractional Laplacian operators. One of the contributions of this work is to show that, while for tempered case the composition yields a tempered fractional Laplacian, the same statement does not hold in the truncated case.

In the second part of the paper we focus on the equivalence between tempered and truncated fractional operators; our second main contribution is an equivalence result for the tempered and truncated energies. In particular, we show that for a given tempered parameter, the associated nonlocal energy norm is equivalent to the truncated energy norm for every truncation parameter.

This paper is organized as follows. In Section 2 we report relevant definitions and

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results that will be used throughout the paper. Section 3 introduces the tempered fractional Laplacian as a special case of a general nonlocal operator by using the nonlocal equivalence kernel from [3]. This is followed by Section 4 where we examine the same problem for the truncated fractional Laplacian. In Section 5, we investigate the relationship between the tempered and truncated fractional operators and prove the equivalence of the corresponding energies. Finally, in Section 6, we summarize our theoretical findings.

2. Notation and previous work. In this section we recall the definitions of unweighted and weighted nonlocal operators and the main result that will be useful throughout the paper.

We let $\Omega \in \mathbb{R}^n$ be an open bounded domain and define the corresponding *interaction domain* as

$$\begin{aligned}\Omega_I &= \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega \text{ such that } \mathbf{x} \text{ interacts with } \mathbf{y} \text{ for some } \mathbf{x} \in \Omega\} \\ &= \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega : |\mathbf{x} - \mathbf{y}| \le \delta \text{ for some } \mathbf{x} \in \Omega\},\end{aligned}$$

where $\delta > 0$ is the so-called interaction radius or horizon. We point out that for fractional operators, including tempered fractional operators, $\delta = \infty$, so that $\Omega_I = \mathbb{R}^n \setminus \Omega$ (see the following section for a precise definition). Let $\mathbf{v} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}$, and let $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be an antisymmetric function such that $supp(\boldsymbol{\alpha}) = B_\delta(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$, where $B_\delta(\mathbf{x})$ is the Euclidean ball of radius δ centered in $\mathbf{x}$. Then, for $\mathbf{x} \in \Omega$ the *nonlocal unweighted divergence and gradient* are defined as

$$\mathcal{D}\mathbf{v}(\mathbf{x}) := \int_{\Omega\cup\Omega_I} (\mathbf{v}(\mathbf{x},\mathbf{y}) + \mathbf{v}(\mathbf{y},\mathbf{x})) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) d\mathbf{y} \tag{2.1}$$

$$\mathcal{G}u(\mathbf{x},\mathbf{y}) := (u(\mathbf{y}) - u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}). \tag{2.2}$$

The nonlocal unweighted Laplacian is obtained by composing the divergence and gradient operators, i.e. for $\mathbf{x} \in \Omega$ and $\gamma = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$ the *nonlocal unweighted Laplacian* is defined as

$$\mathcal{L} = \mathcal{D}\mathcal{G}u(\mathbf{x}) = 2\int_{\Omega\cup\Omega_I} (u(\mathbf{y}) - u(\mathbf{x}))\gamma(\mathbf{x},\mathbf{y}) d\mathbf{y}.$$

Next, we let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric two-point weight function; the *weighted nonlocal divergence and gradient* are defined as

$$\mathcal{D}_\omega \mathbf{v}(\mathbf{x}) := \mathcal{D}(\omega(\mathbf{x},\mathbf{y})\mathbf{v}(\mathbf{x})) = \int_{\Omega\cup\Omega_I} (\omega(\mathbf{x},\mathbf{y})\mathbf{v}(\mathbf{x}) + \omega(\mathbf{y},\mathbf{x})\mathbf{v}(\mathbf{y})) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) d\mathbf{y} \tag{2.3}$$

$$\mathcal{G}_\omega u(\mathbf{x}) := \int_{\Omega\cup\Omega_I} \mathcal{G}u(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) d\mathbf{y} = \int_{\Omega\cup\Omega_I} (u(\mathbf{y}) - u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) d\mathbf{y}. \tag{2.4}$$

As above, for $\mathbf{x} \in \Omega$, by composing the weighted nonlocal divergence and gradient we obtain the *nonlocal weighted Laplacian*

$$\begin{aligned}\mathcal{L}_\omega u(\mathbf{x}) &= \mathcal{D}_\omega \mathcal{G}_\omega u(\mathbf{x}) \\ &= \int_{\Omega\cup\Omega_I} \Bigg[\int_{\Omega\cup\Omega_I} (u(\mathbf{z}) - u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) d\mathbf{z} \\ &\qquad + \int_{\Omega\cup\Omega_I} (u(\mathbf{z}) - u(\mathbf{y}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z}) d\mathbf{z} \Bigg] \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) d\mathbf{y}.\end{aligned}$$

Theorem 4.1 in [3] shows that, given ω and $\boldsymbol{\alpha}$, it is possible to define the so-called *equivalence kernel*, γ_{eq}, for which the composition of weighted divergence and gradient equals the unweighted nonlocal Laplacian with kernel γ_{eq}. We report such result below.

THEOREM 2.1. *Let $\mathcal{D}_\omega$ and $\mathcal{G}_\omega$ be the operators associated with the symmetric weight function ω and the anti-symmetric function $\boldsymbol{\alpha}$. For the equivalence kernel γ_{eq} defined by*

$$\begin{aligned} 2\gamma_{eq}(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} & [\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) \\ & + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \\ & + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})]d\mathbf{z}, \end{aligned} \tag{2.5}$$

the weighted Laplacian $\mathcal{L}_\omega = \mathcal{D}_\omega\mathcal{G}_\omega$ and unweighted Laplacian operator $\mathcal{L}$ with kernel γ_{eq} are equivalent, i.e. $\mathcal{L} = \mathcal{L}_\omega$.

3. Tempered Fractional Laplacian as a Special Case of Nonlocal Operators.

In this section we first show that for a specific choice of ω and $\boldsymbol{\alpha}$ the equivalence kernel is equivalent to the tempered fractional Laplacian kernel and then provide numerical illustrations that confirm the theoretical result. Throughout this section, we assume $u \in H^s(\mathbb{R}^n)$.

3.1. Consistency of tempered fractional Laplacian.

The *tempered fractional Laplacian*, introduced in [9], is defined by

$$\mathcal{L}_{tem}u(\mathbf{x}) := \int_{\Omega\cup\Omega_I} (u(\mathbf{y}) - u(\mathbf{x}))\frac{e^{-\lambda|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{n+2s}}d\mathbf{y} \tag{3.1}$$

where $\lambda > 0$ and $0 < s < 1$ and where $\Omega \cup \Omega_I = \mathbb{R}^n$. Note that we do not consider a scaling constant (which usually appears in the literature for normalization purposes) as it is not relevant for the results reported in this paper. Also, while the integral above should be considered in a principal value sense, we do not explicitly write it in the definition of the operator, and implicitly assume it. Paper [3], shows that for

$$\begin{aligned} \omega(\mathbf{x},\mathbf{y}) &= |\mathbf{y}-\mathbf{x}|\phi(|\mathbf{y}-\mathbf{x}|) \ \text{ with } \ \phi(|\mathbf{y}-\mathbf{x}|) = \frac{e^{-\lambda|\mathbf{x}-\mathbf{y}|}}{|\mathbf{y}-\mathbf{x}|^{n+1+s}} \\ \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) &= \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} \end{aligned} \tag{3.2}$$

the equivalence kernel is given by

$$\gamma_{eq}(\mathbf{x},\mathbf{y}) = \frac{F(n,s,\lambda,|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \tag{3.3}$$

for

$$F(n,s,\lambda,|\mathbf{x}-\mathbf{y}|) = \int_{\mathbb{R}^n} \frac{\mathbf{e}-\mathbf{z}}{|\mathbf{e}-\mathbf{z}|^{n+s+1}}\cdot\frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}}e^{-\lambda|\mathbf{x}-\mathbf{y}|(|\mathbf{e}-\mathbf{z}|+|\mathbf{z}|)}d\mathbf{z}. \tag{3.4}$$

In what follows, we make progress on the following conjecture, stated in [3] as Conjecture 4.1 for the special case of dimension $n = 1$.

CONJECTURE 3.1. *For the function F defined above, there exist positive constants $\underline{C}$ and $\overline{C}$ such that*

$$\underline{C}e^{-\lambda|\mathbf{x}-\mathbf{y}|} \le F(n,s,\lambda,|\mathbf{x}-\mathbf{y}|) \le \overline{C}e^{-\lambda|\mathbf{x}-\mathbf{y}|}. \tag{3.5}$$

Below, Lemma 3.2 establishes the lower bound for $n = 1$. Lemma 3.3 then proves a slightly weaker upper bound for the case $n = 1$, showing that the desired result holds for any $\lambda' > \lambda$. The latter lemma is shown using a linear approximation to the integrand in the conjecture. By using the same strategy and performing a more accurate approximation, it may be possible to prove the conjectured upper bound; we leave this to a future work.

Lemma 3.2. *For the function F defined in (3.4), there exists a constant $\underline{C}$ such that $\underline{C}e^{-\lambda|\mathbf{x}-\mathbf{y}|} \leq F(n, s, \lambda, |\mathbf{x}-\mathbf{y}|)$. In particular,*

$$\underline{C} = \int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz.$$

Proof. For $n = 1$, we analyze

$$F(n, s, \lambda, |x-y|) = \int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|(|1-z|+|z|)} dz.$$

Note that that the non-tempered integral

$$\int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz$$

is a positive number [3]. The integrand of $F(n, s, \lambda, |x-y|)$ is nonnegative for $0 \leq z \leq 1$

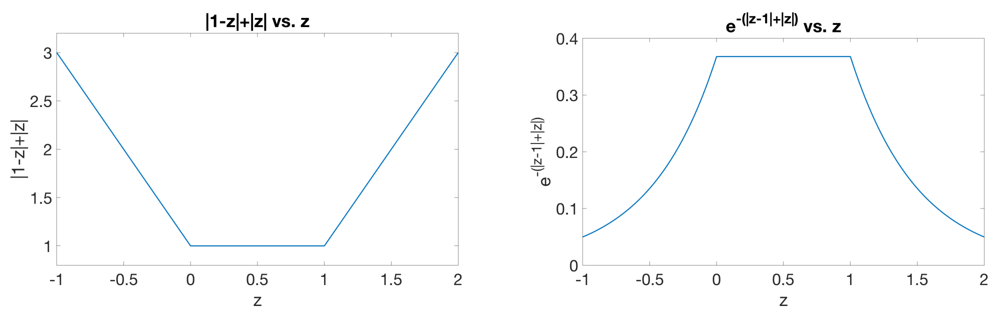

Fig. 3.1: **Left:** Plot of the function $f(z) = |z-1| + |z|$. **Right:** Plot of the exponential function $g(z) = e^{-\lambda|\mathbf{x}-\mathbf{y}|f(z)}$ for $\lambda|\mathbf{x}-\mathbf{y}| = 1$.

and negative elsewhere. We have

$$|1-z| + |z| = \begin{cases} 1-2z, & z < 0 \\ 1, & 0 \leq z \leq 1 \\ 2z-1, & z > 1. \end{cases}$$

Therefore,

$$e^{-\lambda|x-y|(|1-z|+|z|)} = \begin{cases} e^{-\lambda|x-y|(1-2z)} \leq e^{-\lambda|x-y|}, & z < 0 \\ e^{-\lambda|x-y|}, & 0 \leq z \leq 1 \\ e^{-\lambda|x-y|(2z-1)} \leq e^{-\lambda|x-y|}, & z > 1. \end{cases} \tag{3.6}$$

The integrand of $F(n, s, \lambda, |x-y|)$ for $0 \leq z \leq 1$, where the integrand is nonnegative, is

$$e^{-\lambda|x-y|} \int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz.$$

Elsewhere, where the integrand is negative, the upper bounds on the exponential factor provide lower bounds for the integrand in exactly the same from. Thus, we obtain the lower bound

$$F(n, s, \lambda, |x-y|) \geq e^{-\lambda|x-y|} \int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz = Ce^{-\lambda|x-y|}.$$

□

LEMMA 3.3. *For the function F defined in (3.4), for any $\lambda' > \lambda$ there is a constant $\overline{C}$ such that $F(n, s, \lambda, |\mathbf{x}-\mathbf{y}|) \leq \overline{C}e^{-\lambda'|\mathbf{x}-\mathbf{y}|}$.*

Proof. We observe that the factor

$$\frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}}, \tag{3.7}$$

in the integrand of (3.4) is negative if $z < 0$ or $z > 1$ and nonnegative otherwise, while the remaining factor in the integrand is positive. Thus, lower bounds on the factor (3.6) of the integrand for $z < 0$ and $z > 1$ yield upper bounds on the integral (3.4). We claim that

$$e^{-\lambda|x-y|(|1-z|+|z|)} \geq \begin{cases} e^{-\lambda|x-y|}(2|x-y|z+1), & z < 0, \\ e^{-\lambda|x-y|}(2|x-y|(1-z)+1), & z > 1. \end{cases} \tag{3.8}$$

The second inequality follows from the first under the transformation $z \mapsto (1-z)$, which maps $\{z < 0\}$ to $\{z > 1\}$. To prove the first inequality, we denote

$$G(z) = e^{-\lambda|x-y|(|1-z|+|z|)}, \tag{3.9}$$

and note that for $z < 0$,

$$G(z) = e^{-\lambda|x-y|(1-2z)}. \tag{3.10}$$

Then

$$G'(z) = 2|x-y|e^{-\lambda|x-y|(1-2z)}, \tag{3.11}$$

$$G''(z) = 4|x-y|^2 e^{-\lambda|x-y|(1-2z)}. \tag{3.12}$$

Thus $G'(0) = 2|x-y|e^{-\lambda|x-y|}$, and $G''(z) > 0$ for all $z < 0$. Since $G(0) = e^{-\lambda|x-y|}$, it follows that for $z \leq 0$,

$$G(z) \geq G'(0)z + G(0) \tag{3.13}$$

$$\geq 2|x-y|e^{-\lambda|x-y|}z + e^{-\lambda|x-y|} \tag{3.14}$$

$$\geq e^{-\lambda|x-y|}(2|x-y|z+1). \tag{3.15}$$

This proves (3.8). Now define

$$a = \frac{1}{2|x-y|} \geq 0. \tag{3.16}$$

We have, for $z \leq -a$,

$$e^{-\lambda|x-y|}(2|x-y|z+1) \leq 0, \tag{3.17}$$

while for $z \geq 1+a$,

$$e^{-\lambda|x-y|}(2|x-y|(1-z)+1) \leq 0. \tag{3.18}$$

Since $e^{-\lambda|x-y|(|1-z|+|z|)} > 0$ for all z, we can replace (3.8) by

$$e^{-\lambda|x-y|(|1-z|+|z|)} \geq \begin{cases} 0, & z \leq -a, \\ e^{-\lambda|x-y|}(2|x-y|z+1), & -a < z < 0, \\ e^{-\lambda|x-y|}(2|x-y|(1-z)+1), & 1 < z < 1+a, \\ 0, & 1+a \leq z. \end{cases} \tag{3.19}$$

From these inequalities and (3.4), we have

$$\begin{aligned} F(n,s,\lambda,|x-y|) \leq & \int_{-a}^{0} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}(2|x-y|z+1)dz && (3.20) \\ & + \int_{0}^{1} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}dz && (3.21) \\ & + \int_{1}^{1+a} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}(2|x-y|(1-z)+1)dz && (3.22) \\ = & \int_{-a}^{0} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}2|x-y|zdz && (3.23) \\ & + \int_{-a}^{0} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}dz && (3.24) \\ & + \int_{0}^{1} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}dz && (3.25) \\ & + \int_{1}^{1+a} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}dz && (3.26) \\ & + \int_{1}^{1+a} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|}2|x-y|(1-z)dz. && (3.27) \end{aligned}$$

The second, third, and fourth terms above combine to give

$$e^{-\lambda|x-y|} \int_{-a}^{1+a} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz; \tag{3.28}$$

the integral in this expression is convergent due to the convergence of the improper integral

$$\int_{-\infty}^{\infty} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} dz \tag{3.29}$$

proven in [3]. Denote the value of the integral by C, so that the combination (3.28) can be written as $Ce^{-\lambda|x-y|}$. The first term (3.23) can be evaluated as

$$\int_{-a}^{0} \frac{1-z}{|1-z|^{n+s+1}} \cdot \frac{z}{|z|^{n+s+1}} e^{-\lambda|x-y|} 2|x-y| z dz$$
$$= e^{-\lambda|x-y|} 2|x-y| \int_{-a}^{0} \frac{1-z}{(1-z)^{n+s+1}} \cdot \frac{|z|^2}{|z|^{n+s+1}} dz$$
$$= e^{-\lambda|x-y|} 2|x-y| \int_{-a}^{0} (1-z)^{-n-s} |z|^{-n+(1-s)} dz. \quad (3.30)$$

This integral is improper due to the singularity at $z = 0$. Since we can assume that $s < 1$, we have $0 < 1 - s$, so that near $z \approx 0$ the integrand behaves as $z^{-n+\epsilon}$ for $\epsilon > 0$. Therefore, the integral converges and we can write the above as

$$C'|x-y|e^{-\lambda|x-y|} \quad (3.31)$$

for some constant C'. The fifth term (3.27) can be shown to satisfy the same upper bound using a similar calculation. Thus,

$$F(n, s, \lambda, |x-y|) \leq Ce^{-\lambda|x-y|} + C'|x-y|e^{-\lambda|x-y|}. \quad (3.32)$$

In turn, by a continuity and compactness argument, for any λ' there exists a constant $\overline{C}$ such that

$$Ce^{-\lambda|x-y|} + C'|x-y|e^{-\lambda|x-y|} \leq \overline{C}e^{-\lambda'|x-y|}. \quad (3.33)$$

This completes the proof. □

3.2. Numerical illustrations for the tempered fractional Laplacian. The expected behavior for F can be observed in the numerical illustrations presented in this section. Specifically, by displaying values of F in a semilog plot we observe slopes of value $-\lambda$, which indicates that F behaves like $e^{-\lambda|\mathbf{x}-\mathbf{y}|}$.

In Figures 3.2, 3.3, and 3.4, we report such plots for a fixed value for s.

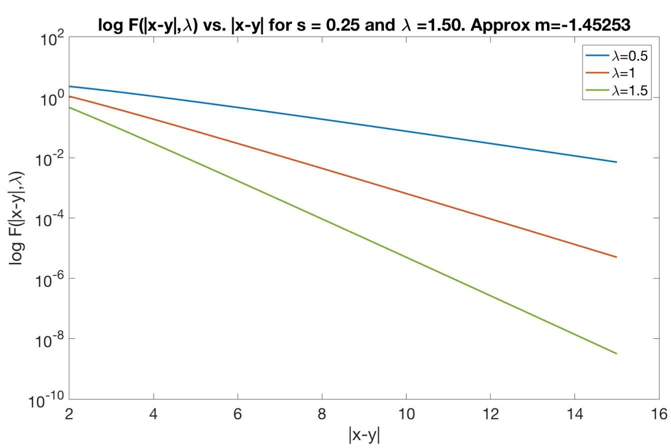

Fig. 3.2: Semilog plot of F vs. $|\mathbf{x}-\mathbf{y}|$ with s=0.25 fixed and varying $\lambda \in \{0.5, 1, 1.5\}$.

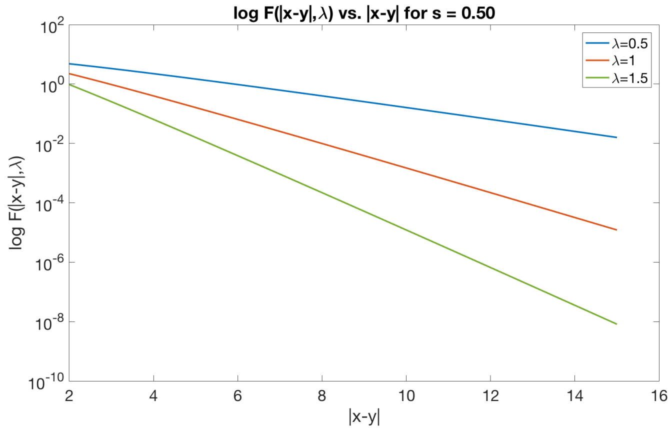

Fig. 3.3: Semilog plot of F vs. $|\mathbf{x}-\mathbf{y}|$ with s=0.5 fixed and varying $\lambda \in \{0.5, 1, 1.5\}$.

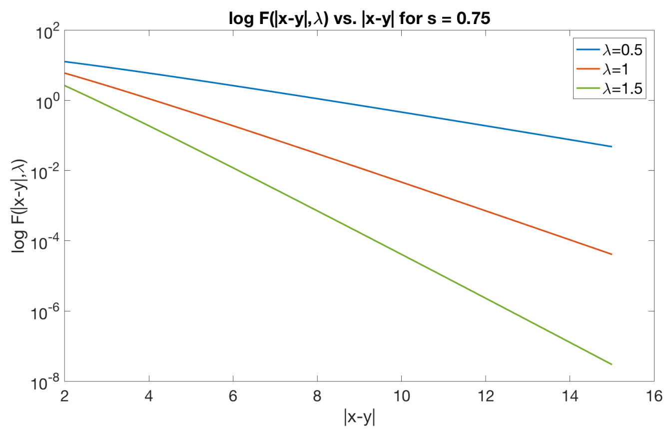

Fig. 3.4: Semilog plot of F vs. $|\mathbf{x}-\mathbf{y}|$ with s=0.75 fixed and varying $\lambda \in \{0.5, 1, 1.5\}$.

4. Truncated Fractional Laplacian as a Special Case of Nonlocal Operators. In this section we proceed as in the previous section and show that, as opposed to tempered operators, the composition of truncated divergence and gradient does not yield the truncated fractional Laplacian. We also provide numerical illustrations that confirm the theoretical result. Throughout this section, we assume $u \in H^s(\mathbb{R}^n)$.

4.1. Lack of equivalence kernel for the truncated fractional Laplacian. For $\mathbf{x} \in \Omega$, we define the *truncated fractional Laplacian* as

$$\mathcal{L}_{tr} u(\mathbf{x}) := \int_{\Omega \cup \Omega_I} (u(\mathbf{y}) - u(\mathbf{x})) \frac{\mathbb{1}\{|\mathbf{y}-\mathbf{x}| \leq \delta\}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y}. \tag{4.1}$$

Also in this case, we do not consider a scaling constant and we implicitly assume that the integral above is intended in a principal value sense.

In [3] the authors show that for a specific choice of $\boldsymbol{\alpha}$ and ω the composition of weighted divergence and gradient yields the fractional Laplacian operator. In this section we consider the same functions and truncate the weight ω by multiplying it by the indicator function $\mathbb{1}\{|\mathbf{y}-\mathbf{x}|<\delta\}$. It is appealing to conjecture that by truncating the weight function over the ball of radius δ, the corresponding composition yields the truncated fractional Laplacian defined above. However, the following result shows that such a conjecture is not true.

THEOREM 4.1. *For $\boldsymbol{\alpha}$ and ω defined as*

$$\omega(\mathbf{x},\mathbf{y}) = |\mathbf{y}-\mathbf{x}|\phi(|\mathbf{y}-\mathbf{x}|) \quad \textit{with} \quad \phi(|\mathbf{y}-\mathbf{x}|) = \frac{\mathbb{1}\{|\mathbf{y}-\mathbf{x}|<\delta\}}{|\mathbf{y}-\mathbf{x}|^{n+1+s}} \tag{4.2}$$

$$\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}, \tag{4.3}$$

the equivalence kernel γ_{eq} has the form

$$\gamma_{eq} = \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2s}} F(|\mathbf{x}-\mathbf{y}|;\delta) \quad \textit{for}$$

$$F(|\mathbf{x}-\mathbf{y}|;\delta) = \int_{\mathbb{R}^n} \frac{\mathbf{e}-\mathbf{z}}{|\mathbf{e}-\mathbf{z}|^{n+s+1}} \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \mathbb{1}\left\{|\mathbf{e}-\mathbf{z}| \le \frac{\delta}{|\mathbf{x}-\mathbf{y}|}\right\} \mathbb{1}\left\{|\mathbf{z}| \le \frac{\delta}{|\mathbf{x}-\mathbf{y}|}\right\} d\mathbf{z}.$$

Proof. With the choices above, we have

$$\begin{aligned}
\gamma_{eq}(\mathbf{x},\mathbf{y}) &= \int_{\mathbb{R}^n} [\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \\
&\qquad + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})]d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \Bigg[\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{n+s+1}} \cdot \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{x}| \le \delta\}\mathbb{1}\{|\mathbf{z}-\mathbf{x}| \le \delta\} \\
&\qquad + \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{y}-\mathbf{x}| \le \delta\} \\
&\qquad + \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}-\mathbf{x}| \le \delta\} \Bigg] d\mathbf{z}.
\end{aligned}$$

We rewrite the expression above as the sum of three terms, $\gamma_{eq}(\mathbf{x},\mathbf{y}) = I + II + III$ for

$$\begin{aligned}
I &= \mathbb{1}\{|\mathbf{y}-\mathbf{x}| \le \delta\} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{n+s+1}} \cdot \int_{\mathbb{R}^n} \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{z}-\mathbf{x}| \le \delta\} d\mathbf{z}, \\
II &= \mathbb{1}\{|\mathbf{y}-\mathbf{x}| \le \delta\} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{n+s+1}} \cdot \int_{\mathbb{R}^n} \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{z}| \le \delta\} d\mathbf{z}, \\
III &= \int_{\mathbb{R}^n} \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}-\mathbf{x}| \le \delta\} d\mathbf{z}.
\end{aligned}$$

Note that $I = II = 0$ due to the rotational symmetry of the integrand. Thus, the truncated kernel is given by

$$\gamma_{eq} = \int_{\mathbb{R}^d} \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{n+s+1}} \mathbb{1}\{|\mathbf{y}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}-\mathbf{x}| \le \delta\} d\mathbf{z}.$$

We can apply the change of variables $\mathbf{z} \mapsto \mathbf{z}+\mathbf{x}$ to obtain

$$\gamma_{eq} = \int_{\mathbb{R}^d} \frac{(\mathbf{y}-\mathbf{x})-\mathbf{z}}{|(\mathbf{y}-\mathbf{x})-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \mathbb{1}\{|(\mathbf{y}-\mathbf{x})-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}| \le \delta\} d\mathbf{z}.$$

This demonstrates that K only depends on $(\mathbf{y}-\mathbf{x})$. Next, we apply a rotation $\mathcal{R}$ and compute

$$\gamma_{eq}\left(\mathcal{R}(\mathbf{x}-\mathbf{y})\right) = \int_{\mathbb{R}^n} \frac{\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathbf{z}}{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \mathbb{1}\{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathbf{z}| \le \delta\} \mathbb{1}\{|\mathbf{z}| \le \delta\} d\mathbf{z}.$$

Let $\mathbf{z} = \mathcal{R}\mathbf{z}_{\text{new}}$. Then $d\mathbf{z} = d\mathbf{z}_{\text{new}}$, and

$$\begin{aligned}
\gamma_{eq}\left(\mathcal{R}(\mathbf{x}-\mathbf{y})\right) &= \int_{\mathbb{R}^n} \frac{\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}_{\text{new}}}{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}_{\text{new}}|^{n+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}_{\text{new}}}{|\mathcal{R}\mathbf{z}_{\text{new}}|^{n+s+1}} \\
&\qquad \mathbb{1}\{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathcal{R}\mathbf{z}| \le \delta\} d\mathbf{z}_{\text{new}} \\
&= \int_{\mathbb{R}^n} \frac{\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}}{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}|^{n+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}}{|\mathcal{R}\mathbf{z}|^{n+s+1}} \\
&\qquad \mathbb{1}\{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathcal{R}\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \frac{\mathcal{R}\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right)}{|\mathcal{R}\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right)|^{n+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}}{|\mathcal{R}\mathbf{z}|^{n+s+1}} \\
&\qquad \mathbb{1}\{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathcal{R}\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathcal{R}\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \frac{1}{|\mathcal{R}\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right)|^{n+s+1}} \frac{1}{|\mathcal{R}\mathbf{z}|^{n+s+1}} \left[\mathcal{R}\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right) \cdot \mathcal{R}\mathbf{z}\right] \\
&\qquad \mathbb{1}\{|\mathcal{R}(\mathbf{y}-\mathbf{x}-\mathbf{z})| \le \delta\}\mathbb{1}\{|\mathcal{R}\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \frac{1}{|\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right)|^{n+s+1}} \frac{1}{|\mathbf{z}|^{n+s+1}} \left[\left((\mathbf{y}-\mathbf{x})-\mathbf{z}\right) \cdot \mathbf{z}\right] \\
&\qquad \mathbb{1}\{|\mathbf{y}-\mathbf{x}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= \int_{\mathbb{R}^n} \frac{(\mathbf{y}-\mathbf{x})-\mathbf{z}}{|(\mathbf{y}-\mathbf{x})-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \\
&\qquad \mathbb{1}\{|\mathbf{y}-\mathbf{x}-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= K(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

This demonstrates that the truncated kernel is a rotationally invariant function of $\mathbf{y}-\mathbf{x}$; that is, it is a function just of the scalar norm of $\mathbf{y}-\mathbf{x}$. Next, we study the scaling by computing

$$\gamma_{eq}(c|\mathbf{x}-\mathbf{y}|;\delta) = \int_{\mathbb{R}^n} \frac{c(\mathbf{y}-\mathbf{x})-\mathbf{z}}{|c(\mathbf{y}-\mathbf{x})-\mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \mathbb{1}\{|c(\mathbf{y}-\mathbf{x})-\mathbf{z}| \le \delta\}\mathbb{1}\{|\mathbf{z}| \le \delta\} d\mathbf{z}.$$

for $c > 0$. Let $\mathbf{z} = c\mathbf{z}_{\text{new}}$. Then $d\mathbf{z} = c^n d\mathbf{z}_{\text{new}}$, and

$$
\begin{aligned}
\gamma_{eq}(c(\mathbf{x}-\mathbf{y});\delta) &= \int_{\mathbb{R}^n} \frac{c(\mathbf{y}-\mathbf{x}) - c\mathbf{z}_{\text{new}}}{|c(\mathbf{y}-\mathbf{x}) - c\mathbf{z}_{\text{new}}|^{n+s+1}} \cdot \frac{c\mathbf{z}_{\text{new}}}{|c\mathbf{z}_{\text{new}}|^{n+s+1}} \\
&\qquad\qquad \mathbb{1}\{|c(\mathbf{y}-\mathbf{x}) - c\mathbf{z}_{\text{new}}| \le \delta\}\mathbb{1}\{|c\mathbf{z}_{\text{new}}| \le \delta\} c^n d\mathbf{z}_{\text{new}} \\
&= \int_{\mathbb{R}^n} \frac{c(\mathbf{y}-\mathbf{x}) - c\mathbf{z}}{|c(\mathbf{y}-\mathbf{x}) - c\mathbf{z}|^{n+s+1}} \cdot \frac{c\mathbf{z}}{|c\mathbf{z}|^{n+s+1}} \\
&\qquad\qquad \mathbb{1}\{c|(\mathbf{y}-\mathbf{x}) - \mathbf{z}| \le \delta\}\mathbb{1}\{c|\mathbf{z}| \le \delta\} c^n d\mathbf{z} \\
&= \frac{c}{c^{n+s+1}} \frac{c}{c^{n+s+1}} c^n \int_{\mathbb{R}^n} \frac{(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \\
&\qquad\qquad \mathbb{1}\{c|(\mathbf{y}-\mathbf{x}) - \mathbf{z}| \le \delta\}\mathbb{1}\{c|\mathbf{z}| \le \delta\} d\mathbf{z} \\
&= \frac{1}{c^{n+2s}} \int_{\mathbb{R}^n} \frac{(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{n+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \\
&\qquad\qquad \mathbb{1}\left\{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}| \le \frac{\delta}{c}\right\} \mathbb{1}\left\{|\mathbf{z}| \le \frac{\delta}{c}\right\} d\mathbf{z}.
\end{aligned}
$$

Then, we write

$$
\begin{aligned}
\gamma_{eq}(|\mathbf{x}-\mathbf{y}|;\delta) &= \gamma_{eq}\left(|\mathbf{x}-\mathbf{y}|\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|};\delta\right) \\
&= K\left(|\mathbf{x}-\mathbf{y}|\mathbf{e};\mathbf{ffi}\right) \\
&= \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \int_{\mathbb{R}^n} \frac{\mathbf{e}-\mathbf{z}}{|\mathbf{e}-\mathbf{z}|^{\mathbf{n+s+1}}} \\
&\qquad \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{n+s+1}} \mathbb{1}\left\{|\mathbf{e}-\mathbf{z}| \le \frac{\mathbf{ffi}}{|\mathbf{x}-\mathbf{y}|}\right\} \mathbb{1}\left\{|\mathbf{z}| \le \frac{\delta}{|\mathbf{x}-\mathbf{y}|}\right\} d\mathbf{z} \\
&= \frac{1}{|\mathbf{x}-\mathbf{y}|^{n+2s}} F(|\mathbf{x}-\mathbf{y}|;\delta).
\end{aligned}
$$

$\square$

By analyzing the form of F, this theorem shows that the equivalence kernel for the truncated fractional weight exhibits unbounded behavior when $|\mathbf{x}-\mathbf{y}| = \delta$ in addition to when $|\mathbf{x}-\mathbf{y}| = 0$, which is not observed in the kernel of the truncated Laplacian. In the next section we illustrate this result using numerical computations of the equivalence kernel in the above theorem.

4.2. Numerical results for the truncated fractional Laplacian. In Figures 4.1 and 4.2 we report the functions F and K above; these plots confirm the singular behavior at $|\mathbf{x}-\mathbf{y}| = \delta$ and confirm that the composition of truncated fractional divergence and gradient is not consistent with the truncated fractional Laplacian.

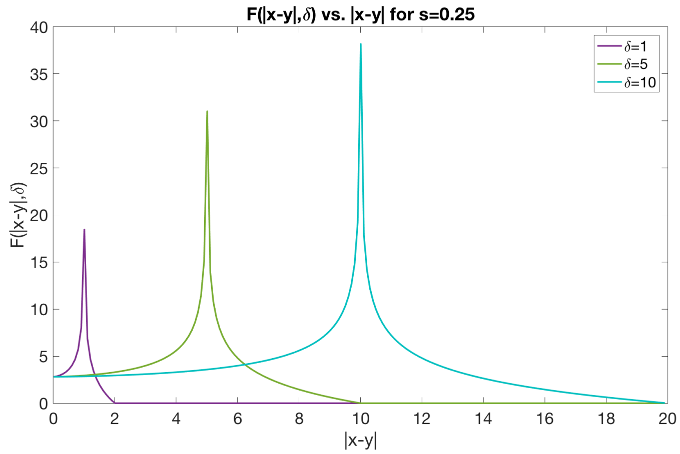

Fig. 4.1: Plot of F vs. $|\mathbf{x}-\mathbf{y}|$ with s=0.25 fixed and varying $\delta \in \{1, 5, 10\}$. Note the singularities at $|\mathbf{x}-\mathbf{y}| = \delta$.

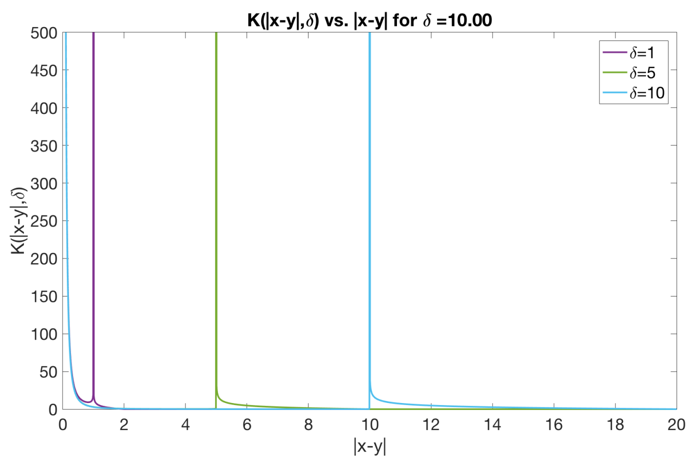

Fig. 4.2: Plot of K vs. $|\mathbf{x}-\mathbf{y}|$ with s=0.25 fixed and varying $\delta \in \{1, 5, 10\}$. While all three equivalence kernels exhibit singularities at $|\mathbf{x}-\mathbf{y}| = 0$ as expected, they are also singular at the respective values of $|\mathbf{x}-\mathbf{y}| = \delta$.

5. Equivalence of Tempered and Truncated Fractional Laplacians. In this section we investigate the relationship between the truncated and tempered fractional Laplacians. Our main goal is to find a viable, but equivalent, alternative to tempered fractional operators, that, due to their infinite interaction range, are extremely computationally expensive. Specifically, we compare the tempered and truncated fractional energy norms and show that given a tempered parameter λ, the associated energy is equivalent to a truncated fractional energy for any truncation parameter δ.

Throughout this section we consider functions $u \in H^s(\mathbb{R}^n)$ such that $u = 0$ in $\mathbb{R}^n \setminus \Omega$ and we refer to this functional space as $H^s_\Omega(\mathbb{R}^n)$. This assumption, though not necessary, simplifies the analysis. For a kernel γ_i, the *nonlocal energy norm* is defined as

$$E_i(u;\mu) = \iint_{(\Omega\cup\Omega_I)^2} (u(\mathbf{x}) - u(\mathbf{y}))^2 \gamma_i(\mathbf{x},\mathbf{y},\mu) d\mathbf{y} d\mathbf{x} \tag{5.1}$$

where μ is a parameter that determines the kernel. We recall that for the tempered and truncated fractional Laplacian operators the kernel γ_i is defined as

$$\gamma_{tem}(\mathbf{x},\mathbf{y},\lambda) = \frac{e^{-\lambda|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} \quad \text{and} \quad \gamma_{tr}(\mathbf{x},\mathbf{y},\delta) = \frac{\mathbb{1}\{|\mathbf{x}-\mathbf{y}|<\delta\}}{|\mathbf{x}-\mathbf{y}|^{n+2s}}, \tag{5.2}$$

respectively. We refer to the corresponding energy norms $E_{tem}(u;\lambda)$ and $E_{tr}(u;\delta)$ as the tempered and truncated energies. Furthermore, by definition of the interaction domain Ω_I, for the tempered fractional Laplacian $\Omega_I = \mathbb{R}^n \setminus \Omega$, whereas for the truncated fractional Laplacian $\Omega_I = \{\mathbf{y} \in \mathbb{R}^n \setminus \Omega : \; |\mathbf{x}-\mathbf{y}| \le \delta, \text{ for some } \mathbf{x} \in \Omega\}$.

In what follows, we show that there exist positive constants $\underline{A}$ and $\overline{A}$ such that, given $\lambda > 0$,

$$\underline{A} E_{tr}(u;\delta) \le E_{tem}(u;\lambda) \le \overline{A} E_{tr}(u;\delta), \quad \forall\, u \in H^s_\Omega(\mathbb{R}^n),\, \delta < \infty. \tag{5.3}$$

The following theorem provides an estimate for the left-hand side of the inequality above.

THEOREM 5.1. *For the nonlocal truncated and tempered energies, the left-hand side of* (5.3) *holds with* $\underline{A} = e^{-\lambda\delta}$.

Proof. Due to the positivity of the integrand,

$$\begin{aligned} E_{tr}(u;\delta) &= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))^2 \gamma_{tr}(\mathbf{x},\mathbf{y},\delta) d\mathbf{y} d\mathbf{x} \\ &= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))^2 \frac{\chi\{|\mathbf{x}-\mathbf{y}|<\delta\}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y} d\mathbf{x} \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x}) - u(\mathbf{y}))^2 \frac{\chi\{|\mathbf{x}-\mathbf{y}|<\delta\}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y} d\mathbf{x}. \end{aligned}$$

Note that for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$,

$$e^{\lambda\delta} e^{-\lambda|\mathbf{x}-\mathbf{y}|} \ge \chi\{|\mathbf{x}-\mathbf{y}|<\delta\}.$$

Thus,

$$\begin{aligned} E_{tr}(u;\delta) &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x}) - u(\mathbf{y}))^2 \frac{\chi\{|\mathbf{x}-\mathbf{y}|<\delta\}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y} d\mathbf{x} \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x}) - u(\mathbf{y}))^2 \frac{e^{\lambda\delta} e^{-\lambda|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y} d\mathbf{x} \\ &= e^{\lambda\delta} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x}) - u(\mathbf{y}))^2 \frac{e^{-\lambda|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y} d\mathbf{x} \\ &= e^{\lambda\delta} E_{tem}(u;\lambda). \end{aligned}$$

□

In the remainder of this section we provide several results yielding the estimate on the right-hand side of inequality (5.3).

The following lemma shows that the integration domain of the truncated energy can be extended to $(\mathbb{R}^n)^2$.

LEMMA 5.2. *For $u \in H^s_\Omega(\mathbb{R}^n)$,*

$$E_{tr}(u;\delta) = \iint_{(\mathbb{R}^n)^2} (u(\mathbf{x}) - u(\mathbf{y}))^2 |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\} d\mathbf{x} d\mathbf{y}.$$

Proof. We let $G = G(\mathbf{x},\mathbf{y}) = (u(\mathbf{x}) - u(\mathbf{y}))^2 |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\}$, consider:

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\, d\mathbf{y} d\mathbf{x} &= \int_{\Omega\cup\Omega_I}\int_{\mathbb{R}^n} G\, d\mathbf{y} d\mathbf{x} + \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I}\int_{\mathbb{R}^n} G\, d\mathbf{y} d\mathbf{x} \\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x} + \int_{\Omega\cup\Omega_I}\int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x} + \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I}\int_{\mathbb{R}^n} G\, d\mathbf{y} d\mathbf{x} \\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x} + \int_{\Omega\cup\Omega_I}\int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x} \\
&\quad + \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x} + \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I}\int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} G\, d\mathbf{y} d\mathbf{x}.
\end{aligned}$$

Here, the last term vanishes because $u \in H^s_\Omega(\mathbb{R}^n)$. For the same reason, and by definition of G, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\, d\mathbf{y} d\mathbf{x} &= E_{tr}(u;\delta) + \int_{\Omega\cup\Omega_I} u^2(\mathbf{x}) \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\} d\mathbf{x} d\mathbf{y} \\
&\quad + \int_{\Omega\cup\Omega_I} u^2(\mathbf{y}) \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\} d\mathbf{x} d\mathbf{y} \\
&= E_{tr}(u;\delta) + \int_{\Omega} u^2(\mathbf{x}) \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\} d\mathbf{x} d\mathbf{y} \\
&\quad + \int_{\Omega} u^2(\mathbf{y}) \int_{\mathbb{R}^n\setminus\Omega\cup\Omega_I} |\mathbf{x}-\mathbf{y}|^{-n-2s} \mathbb{1}\{|\mathbf{x}-\mathbf{y}| \le \delta\} d\mathbf{x} d\mathbf{y} \\
&= E_{tr}(u;\delta).
\end{aligned}$$

Where the last equality follows from that fact that for $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathbb{R}^n\setminus\Omega\cup\Omega_I$, $|\mathbf{x}-\mathbf{y}| > \delta$; i.e. the indicator function is zero. □

The next lemma shows that for every λ there exists some value of the truncation parameter, $\overline{\delta}$ for which the tempered energy is bounded by the truncated energy associated with $\overline{\delta}$.

LEMMA 5.3. *For $\lambda > 0$, there exists a $\overline{\delta} > 0$ independent of u such that*

$$E_{tem}(u;\lambda) \le E_{tr}(u;\overline{\delta}), \quad \forall\, u \in H^s_\Omega(\mathbb{R}^n).$$

Proof. First, note that for any λ, $E_{tem}(u;\lambda) \le E_{tr}(u;\infty)$. Next, we show that there exists $\overline{\delta}$ such that $E_{tr}(u;\infty) \le 2E_{tr}(u;\overline{\delta})$. This result is equivalent to

$$E_{tr}(u;\infty) - E_{tr}(u;\overline{\delta}) \le \frac{1}{2} E_{tr}(u;\infty).$$

Using Lemma 5.2,

$$\begin{aligned} E_{tr}(u;\infty) &- E_{tr}(u;\delta) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x})-u(\mathbf{y}))^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}(1-\mathbb{1}\{|\mathbf{x}-\mathbf{y}|\le\delta\})d\mathbf{x}d\mathbf{y} \\ &\le \delta^{-n-2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x})-u(\mathbf{y}))^2 (1-\mathbb{1}\{|\mathbf{x}-\mathbf{y}|\le\delta\})d\mathbf{x}d\mathbf{y} \\ &\le \delta^{-n-2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y}. \end{aligned}$$

Since $u \equiv 0$ on $\mathbb{R}^n \setminus \Omega$, we have

$$\delta^{-n-2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(\mathbf{x})-u(\mathbf{y}))^2 d\mathbf{x}d\mathbf{y} \le C\delta^{-n-2s}\|u\|^2_{L^2(\Omega)}.$$

By invoking Lemma 4.3 of [4], we have

$$C\delta^{-n-2s}\|u\|^2_{L^2(\Omega)} \le C'\delta^{-n-2s}E_{tr}(u;\infty).$$

We conclude the proof by choosing $\delta = \overline{\delta}$ such that $C'\delta^{-n-2s} \le 1/2$. □

We recall the following result from [2] that state that all truncated energies are equivalent.

Theorem 5.4. *For any $\delta, \delta' > 0$, there exist constants C_1 and C_2 such that*

$$C_1 E_{tr}(u;\delta) \le E_{tr}(u;\delta') \le C_2 E_{tr}(u;\delta).$$

Combining Lemma 5.3 and Theorem 5.4 we obtain the following estimate for the right-hand side of (5.3).

Theorem 5.5. *Given $\lambda > 0$,*

$$E_{tem}(u;\lambda) \le \overline{A}E_{tr}(u;\delta), \quad \forall\, u \in H^s_\Omega(\mathbb{R}^n),$$

where the positive constant $\overline{A}$ depends on Ω and is independent of u.

6. Conclusions. We discussed the consistency of the unified nonlocal Laplacian operator introduced in [3] with the tempered and truncated fractional Laplacian operators via the equivalence kernel. With several numerical tests, we illustrated our theoretical results, confirming that the composition of tempered fractional divergence and gradient yields the tempered fractional Laplacian, whereas the composition of truncated fractional divergence and gradient does not yield, as one might expect, a truncated fractional Laplacian operator.

With the purpose of identifying an operator that is equivalent to the tempered fractional Laplacian, but computationally cheaper, we investigated the relationship between the tempered fractional and truncated fractional energy norms and showed that for a fixed tempered parameter λ, the tempered energy is equivalent to any truncated energy. This result represents a step forward towards the identification of computationally cheap alternatives to fractional operators whose integration domain spans the whole space $\mathbb{R}^n$.

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