TOWARDS A UNIFIED NONLOCAL VECTOR CALCULUS

HAYLEY A. OLSON*, MARTA D'ELIA†, AND MAMIKON GULIAN ‡

Abstract. In this work we provide the groundwork for a unified theory of nonlocal operators. Specifically, in the context of nonlocal diffusion, we prove the equivalence, for certain kernel functions, of weighted and unweighted operators. After studying general properties of the "equivalence" kernel, we show that the equivalence holds for fractional-type operators. We also make preliminary steps towards a unified well-posedness theory that holds for broad class of nonlocal operators by leveraging the well-established theory for unweighted operators, and the generalized operator definition that arises from our equivalence result.

1. Introduction. The use of nonlocal models in place of their classical differential counterparts has been steadily increasing thanks to their potential to capture effects that partial differential equations cannot describe. These effects include multiscale behavior and anomalous behavior such as super- and sub-diffusion and make nonlocal models suitable for a broad class of engineering and scientific applications ranging from fracture mechanics to image processing [1, 2, 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20].

These models are characterized by integral operators, see Fig. 1.1, acting on neighborhoods B_δ (the Euclidean ball of radius δ, referred to as the *horizon*) of size much smaller than the domain or on regions much larger than the domain, including the whole space, see Fig. 1.1 (bottom). The integral form allows one to catch long-range forces and reduces the regularity requirements on the solution. As a result, in a nonlocal model, the state of a system at a point depends on a neighborhood of points. Many challenges arise from modeling and simulation of nonlocal problems, including the non-trivial prescription of boundary conditions, the unresolved treatment of nonlocal interfaces, the uncertainty and sparsity of model parameters and data and the prohibitively high computational cost as the extent of the nonlocal interactions increases; i.e. as the neighborhood becomes larger. Additionally, in the literature we have several independent definitions, formulations, and (possibly incomplete) theories of nonlocal models, see Figure 1.1 for an illustration. Similarities are evident, but they have not been rigorously proved; this is the ultimate goal of this preliminary work.

More specifically, in this paper we determine conditions on the kernel functions γ and η such that unweighted and weighted operators are equivalent. Also, we show that, for a specific choice of η, the weighted nonlocal operator is equivalent to the well-known fractional Laplacian operator. We also make preliminary steps towards a unified well-posedness theory that holds for all classes of operators by leveraging the well-established theory for unweighted operators [10], the generalized definition arising from our equivalence result, and a weighted nonlocal Green's identity [8].

Several reasons make the development of a unified theory impactful. A unified nonlocal vector calculus 1) Connects the nonlocal and fractional communities that would benefit from each other's research; 2) Includes as special cases the well-known classical differential calculus at the limit of vanishing interactions and the fractional calculus at the limit of infinite interactions; 3) Provides the groundwork for *new model discovery* thanks to the broad class of operators that it describes; 4) Describes intrinsically nonlocal phenomena that have not been analyzed or used due to the lack of theory; 5) Guides algorithm, discretization, and solver design.

This paper is organized as follows. In Section 2 we report relevant definitions and results that will be used throughout the paper. In Section 3, we present the derivation

*University of Nebraska - Lincoln, hayley.olson@huskers.unl.edu
†Sandia National Laboratories, mdelia@sandia.gov
‡Sandia National Laboratories, mgulian@sandia.gov

δ-truncated unweighted nonlocal

$$L_\delta u(x) = \int\limits_{B_\delta(x)} (u(x) - u(y))\gamma(x, y)\, dy$$

δ-truncated ω-weighted nonlocal

$$L_\omega u(x) = \int\limits_{B_\delta(x)} (u(x) - u(y)) \int\limits_{\mathbb{R}^n} \eta(y, z; \omega)\, dz\, dy$$

fractional

$$L_s u(x) = \int\limits_{\mathbb{R}^n} (u(x) - u(y)) \frac{C(s, n)}{|x - y|^{n+2s}} dy$$

Fig. 1.1: Classes of nonlocal operators.

of a nonlocal kernel which shows equivalence of the standard nonlocal Laplacian operator with the *weighted* Laplacian operator. This is followed by the analysis of properties of such kernel in Section 4, which includes the equivalence of the weighted nonlocal Laplacian and the fractional Laplacian. Finally, in Section 5, we provide some insights regarding the well-posedness of a class of nonlocal problems.

2. Background and Notation. We introduce weighted and unweighted nonlocal operators following [11]. In particular, let $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2, 3$, be an anti-symmetric vector two-point function. For $\boldsymbol{\nu} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the unweighted nonlocal divergence $\mathcal{D}\boldsymbol{\nu} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathcal{D}\boldsymbol{\nu}(\mathbf{x}) := \int_{\mathbb{R}^n} (\boldsymbol{\nu}(\mathbf{x}, \mathbf{y}) + \boldsymbol{\nu}(\mathbf{y}, \mathbf{x})) \cdot \boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.1}$$

Then for $u : \mathbb{R}^n \to \mathbb{R}$ the unweighted nonlocal gradient, $\mathcal{D}^* u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, negative adjoint of (2.1), is defined as

$$\mathcal{D}^* u(\mathbf{x}, \mathbf{y}) = -(u(\mathbf{y}) - u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{2.2}$$

In this work, as in [11], we consider functions $\boldsymbol{\alpha}$ with bounded support; specifically, we assume that $\boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) = 0$ when $|\mathbf{x} - \mathbf{y}| > \delta$, for some $\delta > 0$. For an open bounded set $\Omega \subset \mathbb{R}^n$ we define the interaction domain Ω_I as the set of points outside of Ω which have a nonzero $\boldsymbol{\alpha}$ interaction with points inside Ω. More specifically[1],

$$\Omega_I = \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega : \boldsymbol{\alpha}(\mathbf{x}, \mathbf{y}) \neq 0, \mathbf{x} \in \Omega\} = \{\mathbf{y} \in \mathbb{R}^n \backslash \Omega : |\mathbf{x} - \mathbf{y}| \leq \delta, \mathbf{x} \in \Omega\}.$$

Note that this set plays the role of *nonlocal boundary*; in fact, when solving nonlocal diffusion equations in Ω, volume constraints on the solution have to be prescribed on Ω_I to guarantee well-posedness.

For the kernel $\gamma = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$ and $\mathbf{x} \in \Omega$ we define the unweighted δ-truncated nonlocal Laplacian as the composition of unweighted nonlocal divergence and gradient, i.e.

$$L_\delta u(\mathbf{x}) = \mathcal{D}\mathcal{G}u(\mathbf{x}) = \int_{B_\delta(\mathbf{x})} (u(\mathbf{x}) - u(\mathbf{y}))\gamma(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \tag{2.3}$$

[1]Note that Ω and Ω_I need not be adjacent, which differs from the boundary of a domain. Also note that if we set $\Omega = \mathbb{R}^n$, then Ω_I is empty.

In [11] the reader can find results regarding well-posedness of equations involving (2.3) and further results such as integration by parts and Green's identities.

Operators (2.1) and (2.2) are the building blocks of weighted nonlocal operators. The major shift between the two is that the weighted operators are one-point functions. Throughout, we let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a non-negative, symmetric scalar function. For $\boldsymbol{\nu} : \mathbb{R}^n \to \mathbb{R}^n$, the weighted nonlocal divergence $\mathcal{D}_\omega \boldsymbol{\nu} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathcal{D}_\omega \boldsymbol{\nu}(\mathbf{x}) := \mathcal{D}(\omega(\mathbf{x},\mathbf{y})\boldsymbol{\nu}(\mathbf{x})) = \int_{\mathbb{R}^n} (\omega(\mathbf{x},\mathbf{y})\boldsymbol{\nu}(\mathbf{x}) + \omega(\mathbf{y},\mathbf{x})\boldsymbol{\nu}(\mathbf{y})) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.4}$$

For $u : \mathbb{R}^n \to \mathbb{R}$, the weighted nonlocal gradient $\mathcal{D}^*_\omega u : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathcal{D}^*_\omega u(\mathbf{x}) := \int_{\mathbb{R}^n} \mathcal{D}^* u(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.5}$$

Paper [11] show that the latter is the negative adjoint of the former. As done in the unweighted case, we define the δ-truncated, ω-weighted nonlocal Laplacian as the composition of (2.4) and (2.5), i.e., for $\mathbf{x} \in \Omega$

$$\begin{aligned} L_\omega u(\mathbf{x}) &= \mathcal{D}_\omega \mathcal{G}_\omega u(\mathbf{x}) = \mathcal{D}_\omega \mathcal{D}^*_\omega u(\mathbf{x}) = \mathcal{D}(\omega(\mathbf{x},\mathbf{y})\mathcal{D}^*_\omega u(\mathbf{x})) \\ &= \int_{\Omega\cup\Omega_I} \Big[\omega(\mathbf{x},\mathbf{y})\mathcal{D}^*_\omega u(\mathbf{x}) + \omega(\mathbf{y},\mathbf{x})\mathcal{D}^*_\omega u(\mathbf{y})\Big] \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{2.6}$$

Properties of this operator and its connection with (2.3) are analyzed in the following section.

The fractional Laplacian operator. A nonlocal operator that is ubiquitous in the literature is the fractional Laplacian $(-\Delta)^s$. For $s \in (0,1)$ and $f : \mathbb{R}^n \to \mathbb{R}$, it is defined as

$$(-\Delta)^s(f)(\mathbf{x}) = c_{n,s} \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+2s}} d\mathbf{y}$$

with

$$c_{n,s} = \frac{4^s \Gamma(n/2 - s)}{\pi^{n/2}|\Gamma(-s)|}.$$

Paper [9] shows that for $\delta \to \infty$, the unweighted δ-truncated nonlocal Laplacian converges to $(-\Delta)^s$; one of our goals is to show that the latter can also be expressed as a composition of ω-weighted, δ-truncated nonlocal Laplacian for a specific choice of $\boldsymbol{\alpha}$ and ω.

3. Equivalence of the weighted and unweighted Laplacian operator. Given the scalar point function u, we want to establish the equivalence of $L_\omega u(\mathbf{x})$ and $L_\delta u(\mathbf{x})$ for some choice of kernel γ. Due to the symmetry of $\omega(\mathbf{x},\mathbf{y})$, we have

$$
\begin{aligned}
\mathcal{D}_\omega \mathcal{G}_\omega u(\mathbf{x}) &= \int_{\Omega\cup\Omega_I} (\mathcal{D}^*_\omega u(\mathbf{x}) + \mathcal{D}^*_\omega u(\mathbf{y}))\cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}\\
&= \int_{\Omega\cup\Omega_I}\Bigg[\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})d\mathbf{z}\\
&+\int_{\Omega\cup\Omega_I}(u(\mathbf{y})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})d\mathbf{z}\Bigg]\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}\\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z} \qquad (3.1)\\
&+ \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{y})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}. \qquad (3.2)
\end{aligned}
$$

Let the integral in (3.1) be A and the one in (3.2) be B. We have

$$
\begin{aligned}
A &= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}\\
&= \int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})\int_{\Omega\cup\Omega_I}\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}.
\end{aligned}
$$

Letting $\gamma_1(\mathbf{x},\mathbf{z}) = \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})\int_{\Omega\cup\Omega_I}\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}$, we have

$$
A = \int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\gamma_1(\mathbf{x},\mathbf{z})d\mathbf{z}.
$$

Next,

$$
\begin{aligned}
B &= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{y})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}\\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{y})-u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}\\
&+ \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}.
\end{aligned}
$$

Switching $\mathbf{y}$ and $\mathbf{z}$ in the first integral, and employing the anti-symmetry of $\boldsymbol{\alpha}$ and symmetry of ω, we find

$$
\begin{aligned}
B &= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{z})-u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})d\mathbf{z}d\mathbf{y}\\
&+ \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}\\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})d\mathbf{z}d\mathbf{y}\\
&+ \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{z}\\
&= \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot[\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})+\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})]d\mathbf{y}d\mathbf{z}\\
&= \int_{\Omega\cup\Omega_I}(u(\mathbf{x})-u(\mathbf{z}))\int_{\Omega\cup\Omega_I}\boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z})\cdot[\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})+\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})]d\mathbf{y}d\mathbf{z}.
\end{aligned}
$$

Letting $\gamma_2(\mathbf{x},\mathbf{z}) = \int_{\Omega\cup\Omega_I} \boldsymbol{\alpha}(\mathbf{y},\mathbf{z})\omega(\mathbf{y},\mathbf{z}) \cdot [\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) + \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})]d\mathbf{y}$ gives us

$$B = \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{z}))\gamma_2(\mathbf{x},\mathbf{z})d\mathbf{z}.$$

By combining the above, we have

$$\begin{aligned}
\mathcal{D}_\omega \mathcal{G}_\omega u(\mathbf{x}) &= A + B \\
&= \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{z}))\gamma_1(\mathbf{x},\mathbf{z})d\mathbf{z} + \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{z}))\gamma_2(\mathbf{x},\mathbf{z})d\mathbf{z} \\
&= \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{z}))(\gamma_1(\mathbf{x},\mathbf{z}) + \gamma_2(\mathbf{x},\mathbf{z}))d\mathbf{z}.
\end{aligned}$$

Thus, for

$$\begin{aligned}
\gamma(\mathbf{x},\mathbf{y}) &= \gamma_1(\mathbf{x},\mathbf{y}) + \gamma_2(\mathbf{x},\mathbf{y}) \\
&= \int_{\Omega\cup\Omega_I} [\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) \qquad (3.3)\\
&\qquad + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})]d\mathbf{z},
\end{aligned}$$

the operators $L_\delta u(\mathbf{x})$ and $L_\omega u(\mathbf{x})$ are equivalent.

4. Properties of the equivalence kernel. In this section we analyze properties of the kernel in (3.3); specifically, we investigate its symmetry and show equivalence of $L_\omega u(\mathbf{x})$ with the well-known fractional Laplacian operator for a specific choice of ω and $\boldsymbol{\alpha}$.

We point out that one of our major goals is to find conditions on the equivalence kernel that guarantee well-posedness of the associated nonlocal diffusion operator. This result, that would enable characterization of a broad class of well-posed nonlocal diffusion problems, is the subject of current research.

4.1. Symmetry of the equivalence kernel. The symmetry of γ can be shown directly using the antisymmetry of $\boldsymbol{\alpha}$ and the symmetry of ω. Throughout this section, we let $\boldsymbol{\eta}(\mathbf{x},\mathbf{y})$ be the antisymmtric function defined as $\boldsymbol{\eta}(\mathbf{x},\mathbf{y}) = \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})$; then, we rewrite (3.3) as

$$\gamma(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} [\boldsymbol{\eta}(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\eta}(\mathbf{x},\mathbf{z}) + \boldsymbol{\eta}(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\eta}(\mathbf{x},\mathbf{y}) + \boldsymbol{\eta}(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\eta}(\mathbf{x},\mathbf{z})]d\mathbf{z}$$

The antisymmetry of $\boldsymbol{\eta}$ implies that

$$\gamma(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} [\boldsymbol{\eta}(\mathbf{y},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{z},\mathbf{x}) + \boldsymbol{\eta}(\mathbf{y},\mathbf{z}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{x}) + \boldsymbol{\eta}(\mathbf{y},\mathbf{z}) \cdot \boldsymbol{\eta}(\mathbf{z},\mathbf{x})]d\mathbf{z}$$

Since the dot product is commutative, we switch the orders of each $\boldsymbol{\eta}$ pair:

$$\gamma(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} [\boldsymbol{\eta}(\mathbf{z},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{x}) + \boldsymbol{\eta}(\mathbf{y},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{z}) + \boldsymbol{\eta}(\mathbf{z},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{z})]d\mathbf{z}$$

then, switching the first two terms, we have

$$\begin{aligned}
\gamma(\mathbf{x},\mathbf{y}) &= \int_{\Omega\cup\Omega_I} [\boldsymbol{\eta}(\mathbf{y},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{z}) + \boldsymbol{\eta}(\mathbf{z},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{x}) + \boldsymbol{\eta}(\mathbf{z},\mathbf{x}) \cdot \boldsymbol{\eta}(\mathbf{y},\mathbf{z})]d\mathbf{z} \\
&= \gamma(\mathbf{y},\mathbf{x}).
\end{aligned}$$

4.2. Equivalence of L_ω and the fractional Laplacian. In this section we show the equivalence of the w-weighted diffusion operator and the fractional Laplacian operator for the following choice of weight and kernel functions:

$$\omega(\mathbf{x},\mathbf{y}) = |\mathbf{y}-\mathbf{x}|\phi(|\mathbf{y}-\mathbf{x}|) \quad \text{for} \quad \phi(|\mathbf{y}-\mathbf{x}|) = \frac{1}{|\mathbf{y}-\mathbf{x}|^{d+1+s}}$$
$$\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|},$$

for which

$$\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) = |\mathbf{y}-\mathbf{x}|\phi(|\mathbf{y}-\mathbf{x}|)\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|} = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}.$$

Thus,

$$\begin{aligned}
\gamma(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} &[\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \\
&+ \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y})\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})]d\mathbf{z} \\
&= \int_{\Omega\cup\Omega_I}\left[\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}\cdot\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}} + \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}\right. \\
&\left. + \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}}\right]d\mathbf{z}.
\end{aligned}$$

We rewrite the expression above as the sum of three terms, $K(\mathbf{x},\mathbf{y}) = I + II + III$:

$$I = \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}\cdot\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}}dz = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}\cdot\int_{\mathbb{R}^d}\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}}dz, \tag{4.1}$$
$$II = \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}dz = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^{d+s+1}}\cdot\int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}dz, \tag{4.2}$$
$$III = \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}}dz. \tag{4.3}$$

Due to the rotational symmetry of the domain of integration, the integrals in I and II are both zero. Thus, the kernel is just the term III, that we rename K:

$$K(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{z}-\mathbf{x}}{|\mathbf{z}-\mathbf{x}|^{d+s+1}}d\mathbf{z}. \tag{4.4}$$

We evaluate this integral indirectly. Let $\mathbf{z}_{\text{new}} = \mathbf{z}-\mathbf{x}$. Then $\mathbf{z} = \mathbf{z}_{\text{new}} + \mathbf{x}$, $d\mathbf{z} = d\mathbf{z}_{\text{new}}$ and $\mathbf{y}-\mathbf{z} = \mathbf{y}-\mathbf{z}_{\text{new}}-\mathbf{x} = \mathbf{y}-\mathbf{x}-\mathbf{z}_{\text{new}}$. Thus,

$$K(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{x}-\mathbf{z}_{\text{new}}}{|\mathbf{y}-\mathbf{x}-\mathbf{z}_{\text{new}}|^{d+s+1}}\cdot\frac{\mathbf{z}_{\text{new}}}{|\mathbf{z}_{\text{new}}|^{d+s+1}}d\mathbf{z}_{\text{new}} \tag{4.5}$$
$$= \int_{\mathbb{R}^d}\frac{\mathbf{y}-\mathbf{x}-\mathbf{z}}{|\mathbf{y}-\mathbf{x}-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}}d\mathbf{z}. \tag{4.6}$$

From this, it follows that $K(\mathbf{x},\mathbf{y})$ depends only on $\mathbf{x}-\mathbf{y}$, i.e., we can write $K(\mathbf{x},\mathbf{y}) = K(\mathbf{x}-\mathbf{y})$. Next, we show that $K(\mathbf{x}-\mathbf{y})$ is rotationally invariant. Consider a rotation $\mathcal{R}$; we have

$$K\left(\mathcal{R}(\mathbf{x}-\mathbf{y})\right) = \int_{\mathbb{R}^d}\frac{\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathbf{z}}{|\mathcal{R}(\mathbf{y}-\mathbf{x})-\mathbf{z}|^{d+s+1}}\cdot\frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}}d\mathbf{z}. \tag{4.7}$$

Let $\mathbf{z} = \mathcal{R}\mathbf{z}_{\text{new}}$. Then $d\mathbf{z} = d\mathbf{z}_{\text{new}}$, and

$$K\left(\mathcal{R}(\mathbf{x}-\mathbf{y})\right) = \int_{\mathbb{R}^d} \frac{\mathcal{R}(\mathbf{y}-\mathbf{x}) - \mathcal{R}\mathbf{z}_{\text{new}}}{|\mathcal{R}(\mathbf{y}-\mathbf{x}) - \mathcal{R}\mathbf{z}_{\text{new}}|^{d+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}_{\text{new}}}{|\mathcal{R}\mathbf{z}_{\text{new}}|^{d+s+1}} d\mathbf{z}_{\text{new}} \tag{4.8}$$

$$= \int_{\mathbb{R}^d} \frac{\mathcal{R}(\mathbf{y}-\mathbf{x}) - \mathcal{R}\mathbf{z}}{|\mathcal{R}(\mathbf{y}-\mathbf{x}) - \mathcal{R}\mathbf{z}|^{d+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}}{|\mathcal{R}\mathbf{z}|^{d+s+1}} d\mathbf{z} \tag{4.9}$$

$$= \int_{\mathbb{R}^d} \frac{\mathcal{R}\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right)}{|\mathcal{R}\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right)|^{d+s+1}} \cdot \frac{\mathcal{R}\mathbf{z}}{|\mathcal{R}\mathbf{z}|^{d+s+1}} d\mathbf{z} \tag{4.10}$$

$$= \int_{\mathbb{R}^d} \frac{1}{|\mathcal{R}\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right)|^{d+s+1}} \frac{1}{|\mathcal{R}\mathbf{z}|^{d+s+1}} \left[\mathcal{R}\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right) \cdot \mathcal{R}\mathbf{z}\right] \; d\mathbf{z} \tag{4.11}$$

$$= \int_{\mathbb{R}^d} \frac{1}{|\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right)|^{d+s+1}} \frac{1}{|\mathbf{z}|^{d+s+1}} \left[\left((\mathbf{y}-\mathbf{x}) - \mathbf{z}\right) \cdot \mathbf{z}\right] \; d\mathbf{z} \tag{4.12}$$

$$= \int_{\mathbb{R}^d} \frac{(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{d+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}} \; d\mathbf{z} \tag{4.13}$$

$$= K(\mathbf{x}-\mathbf{y}). \tag{4.14}$$

Therefore, K depends only on $|\mathbf{x}-\mathbf{y}|$ and we write $K(\mathbf{x},\mathbf{y}) = K(|\mathbf{x}-\mathbf{y}|)$. Now we let $\lambda > 0$ and consider

$$K(\lambda|\mathbf{x}-\mathbf{y}|) = \int_{\mathbb{R}^d} \frac{\lambda(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|\lambda(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{d+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}} d\mathbf{z}. \tag{4.15}$$

Let $\mathbf{z} = \lambda\mathbf{z}_{\text{new}}$. Then $d\mathbf{z} = \lambda^d d\mathbf{z}_{\text{new}}$, and

$$K(\lambda|\mathbf{x}-\mathbf{y}|) = \int_{\mathbb{R}^d} \frac{\lambda(\mathbf{y}-\mathbf{x}) - \lambda\mathbf{z}_{\text{new}}}{|\lambda(\mathbf{y}-\mathbf{x}) - \lambda\mathbf{z}_{\text{new}}|^{d+s+1}} \cdot \frac{\lambda\mathbf{z}_{\text{new}}}{|\lambda\mathbf{z}_{\text{new}}|^{d+s+1}} \lambda^d d\mathbf{z}_{\text{new}} \tag{4.16}$$

$$= \int_{\mathbb{R}^d} \frac{\lambda(\mathbf{y}-\mathbf{x}) - \lambda\mathbf{z}}{|\lambda(\mathbf{y}-\mathbf{x}) - \lambda\mathbf{z}|^{d+s+1}} \cdot \frac{\lambda\mathbf{z}}{|\lambda\mathbf{z}|^{d+s+1}} \lambda^d dz \tag{4.17}$$

$$= \frac{\lambda}{\lambda^{d+s+1}} \frac{\lambda}{\lambda^{d+s+1}} \lambda^d \int_{\mathbb{R}^d} \frac{(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{d+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}} dz \tag{4.18}$$

$$= \frac{1}{\lambda^{d+2s}} \int_{\mathbb{R}^d} \frac{(\mathbf{y}-\mathbf{x}) - \mathbf{z}}{|(\mathbf{y}-\mathbf{x}) - \mathbf{z}|^{d+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}} dz \tag{4.19}$$

$$= \frac{1}{\lambda^{d+2s}} K(|\mathbf{x}-\mathbf{y}|). \tag{4.20}$$

So, we can say that

$$K(\mathbf{x}-\mathbf{y}) = \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2s}} K(\mathbf{e}), \tag{4.21}$$

where $\mathbf{e}$ is any unit vector and $K(\mathbf{e})$ is a constant, independent of the choice of $\mathbf{e}$ (since K is rotationally invariant). Thus we have

$$K(\mathbf{e}) = \int_{\mathbb{R}^d} \frac{\mathbf{e} - \mathbf{z}}{|\mathbf{e} - \mathbf{z}|^{d+s+1}} \cdot \frac{\mathbf{z}}{|\mathbf{z}|^{d+s+1}} d\mathbf{z}. \tag{4.22}$$

In one dimension, we proved that $K(\mathbf{e})$ is a positive constant; this is confirmed by numerical tests.

5. Implications on the well-posedness. Paper [12] proves that, under certain conditions on the kernel function γ, the operator L_δ is associated with a coercive variational form, or, in other words, with an energy norm. This, in turn, provides well-posedness of the following diffusion problem:

$$\begin{cases} -L_\delta(u) = f & \text{in } \Omega \\ u = g & \text{in } \Omega_I. \end{cases}$$

Utilizing the above equivalence of $\mathcal{D}\mathcal{G}$ and $\mathcal{D}_\omega\mathcal{G}_\omega$ along with the nonlocal Green's identity for weighted operators [8], we can show that the energy norm associated with the unweighted nonlocal operators is equivalent to that of weighted operators, thus providing well-posedness of diffusion problems such as

$$\begin{cases} -L_\omega(u) = f & \text{in } \Omega \\ u = g & \text{in } \Omega_I. \end{cases}$$

More specifically, for a symmetric kernel γ, the unweighted energy norm is defined as

$$|||u|||_\delta^2 = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))^2 \gamma(\mathbf{x},\mathbf{y}) d\mathbf{y} d\mathbf{x}.$$

On the other hand, the weighted energy norm is defined as

$$|||u|||_\omega^2 = \int_{\Omega\cup\Omega_I} (\mathcal{D}_\omega u(\mathbf{x}))^2 d\mathbf{x}.$$

By applying the weighted nonlocal Green's identity [8] and defining γ as in (3.3), it is easy to show that $|||u|||_\delta = |||u|||_\omega$.

The extension of this equivalence to a broad class of nonlocal operators is the subject of our current research.

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